

Box spaces of the free group that neither contain expanders nor embed into a Hilbert space

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Abstract

We construct box spaces of a free group that do not coarsely embed into a Hilbert space, but do not contain coarsely embedded expanders. We do this by considering two sequences of subgroups of the free group: one which gives rise to a box space which forms an expander, and another which gives rise to a box space that can be coarsely embedded into a Hilbert space. We then take certain intersections of these subgroups, and prove that the corresponding box space contains generalized expanders. We show that there are no coarsely embedded expanders in the box space corresponding to our chosen sequence by proving that a box space that covers another box space of the same group that is coarsely embeddable into a Hilbert space cannot contain coarsely embedded expanders.

1 Introduction

Given a residually finite, finitely generated group G , we say that a sequence of nested finite index normal subgroups of the group is a *filtration* if this sequence of subgroups has trivial intersection. Given such a filtration $\{N_i\}$ of G and fixing a generating set of G , we can consider each finite quotient G/N_i with the Cayley graph metric induced by image the generating set of G .

Definition 1.1. *The box space $\square_{N_i} G$ of G with respect to a filtration $\{N_i\}$ is the disjoint union of the finite quotients $\{G/N_i\}$ with their Cayley graph metrics, with the distance between different quotients defined to be at least the larger of their diameters.*

Note that different choices of generating set for G give rise to coarsely equivalent box spaces ([Kh12]). Studying the geometric properties of this space essentially reduces to studying geometric properties that the finite quotients G/N_i have *uniformly*. To give an example, let us first introduce the following notion.

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Definition 1.2. Given two metric spaces (X, d_X) and (Y, d_Y) , we say that a map $f : X \rightarrow Y$ is a coarse embedding if there exist non-decreasing control functions $\rho_{\pm} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $a, b \in X$, we have

$$\rho_-(d_X(a, b)) \leq d_Y(f(a), f(b)) \leq \rho_+(d_X(a, b)).$$

In this way, a box space coarsely embeds into a Hilbert space if and only if all of the components G/N_i admit coarse embeddings into a Hilbert space with the same control functions ρ_{\pm} .

Box spaces are good sources of examples in the world of bounded geometry metric spaces (i.e. ones in which the cardinality of balls is uniformly bounded by some constant depending only on the radius) because their geometric properties often have strong links with algebraic or analytic properties of the parent group G . For example, the first explicit construction of expander graphs, due to Margulis ([Mar]), was given in the form of a box space of a group with Kazhdan's property (T). Expanders are sequences of graphs of bounded degree which are highly connected. These seemingly contradictory properties of sparseness and connectivity make them sought-after objects in computer science, network design, and computational group theory, as well as interesting geometric objects in their own right. The connectivity properties of expanders are also what prevents them from embedding well into Banach spaces. For a long time, the presence of coarsely embedded expanders was in fact the only known obstruction to a bounded geometry metric space coarsely embedding into a Hilbert space. Note that if one does not impose the condition of bounded geometry, then ℓ^p with $p > 2$ is a space which does not contain coarsely embedded expanders, yet does not admit an embedding into a Hilbert space ([JR]).

An important step towards answering the question of whether expanders are indeed the only possible obstruction was the paper of Tessera [Tes], in which he was able to give a characterization of spaces which do not embed coarsely into a Hilbert space in terms of *generalized expanders*, which satisfy corresponding Poincaré inequalities relative to a measure.

In the groundbreaking article [AT], Arzhantseva and Tessera gave examples of sequences of finite Cayley graphs of uniformly bounded degree which do not contain coarsely embedded expanders but do not embed coarsely into a Hilbert space. Their examples make use of *relative expanders*, which are a specific case of generalized expanders. One of their examples is a box space of $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$, a group with relative property (T): this box space does not embed into a Hilbert space because the parent group does not have the Haagerup property, and it does not contain expanders thanks to a proposition (Proposition 2, [AT]) which shows that expanders cannot be embedded into a sequence of group extensions where the sequence of quotients and the sequence of normal subgroups which make up the extension both embed coarsely into a Hilbert space. They also give constructions of box spaces of wreath products, including an example which admits a fibred coarse embedding into a Hilbert space (i.e. it is a box space of a group with the Haagerup property, see [CWW] for the proof of this equivalence). All of these examples are constructed using sequences of finite groups which do embed into a Hilbert space, and the non-embeddability of the resulting spaces is encoded in the action of one subgroup on another.

The following problem ([AT], Section 8: Open Problems) remained open: does there exist a sequence of finite graphs with bounded degree and girth (i.e. the length of the smallest cycle) tending to infinity, which does not coarsely embed into a Hilbert space but does not contain a coarsely embedded expander? A natural way to construct such a sequence would be to use a box space of a non-abelian free group. This requires a different method to the one used in [AT], since

there are no obvious “building blocks” which can be used to construct the sequence (as with the semidirect products of embeddable groups in [AT]).

In this article, we answer this question by proving the following theorem.

Main Theorem. *There exists a filtration of the free group F_3 such that the corresponding box space does not coarsely embed into a Hilbert space, but does not admit a coarsely embedded expander sequence.*

The proof of the theorem involves taking a sequence of subgroups which gives rise to an embeddable box space, and intersecting it with one which gives rise to an expander, in a controlled way.

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Overview

The overall structure of the proof is as follows. We construct a sequence of subgroups $\{N_i\}$ of F_3 which gives rise to expanders (Section 3.1), and consider the sequence of homology covers of the quotients $\{F_3/N_i\}$; this gives rise to another sequence of subgroups $\Gamma(N_i) < N_i$ of F_3 (Section 3.2) such that the corresponding quotients of F_3 coarsely embed into a Hilbert space. We then consider the quotients of F_3 by intersections of these sequences of subgroups, as in the following diagram, where the arrows represent quotient maps.

$$\begin{array}{ccccccc}
 & & & & & & F_3/\Gamma(N_3) \cdots \\
 & & & & & & \downarrow \\
 & & & & & F_3/\Gamma(N_2) \longleftarrow & F_3/(N_3 \cap \Gamma(N_2)) \cdots \\
 & & & & \downarrow & & \downarrow \\
 F_3/\Gamma(N_1) \longleftarrow & F_3/(N_2 \cap \Gamma(N_1)) \longleftarrow & F_3/(N_3 \cap \Gamma(N_1)) \cdots \\
 \downarrow & \downarrow & \downarrow \\
 \{1\} \longleftarrow & F_3/N_1 \longleftarrow & F_3/N_2 \longleftarrow & F_3/N_3 \cdots
 \end{array}$$

In Section 4, we choose a subsequence of quotients $\{F_3/(N_{n_i} \cap \Gamma(N_{k_i}))\}$ which lie on some path that moves sufficiently slowly away the horizontal (expander) sequence in this “triangle” of intersections.

We do this so that for such a quotient $F_3/(N_{n_i} \cap \Gamma(N_{k_i}))$, we can control the eigenvalues corresponding to those eigenvectors of the Laplacian which are not coming from lifts of eigenvectors of the Laplacian on the quotient $F_3/(N_{n_i-1} \cap \Gamma(N_{k_i}))$ which is horizontally to the left of $F_3/(N_{n_i} \cap \Gamma(N_{k_i}))$ (we do this using representation theory in Section 3.3). This ensures, via the results on generalized expanders of Section 2.1 that the chosen sequence will not coarsely embed into a Hilbert space.

On the other hand, each of the quotients $F_3/(N_{n_i} \cap \Gamma(N_{k_i}))$ surjects onto $F_3/\Gamma(N_{k_i})$, and we prove in Section 2.2 that such a sequence then cannot contain coarsely embedded expanders.

2 Expanders and embeddability into Hilbert spaces

2.1 Expanders and generalized expanders

Let $X = (E, V)$ be a finite, k -regular graph, and number the vertices of X , $V = \{v_1, v_2, \dots, v_n\}$. The *adjacency matrix* of X is the matrix A indexed by pairs of vertices $v_i, v_j \in V$ such that A_{ij} is equal to the number of edges connecting v_i to v_j . We will restrict ourselves to considering simple graphs, and so for us, this number will always be equal to either 0 or 1.

The *Laplacian* is defined as the matrix $\Delta := k\text{Id} - A$, which can be viewed as an operator $\ell^2(V) \rightarrow \ell^2(V)$. If $|V| = n$, then Δ is an $n \times n$ symmetric matrix and thus, counting multiplicities, has n real eigenvalues,

$$\lambda_0 = 0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}.$$

Note that the corresponding eigenvectors are orthogonal. The first non-trivial eigenvalue λ_1 is linked to connectivity properties of the graph X , namely via the *Cheeger constant* $h(X) := \inf |\partial F|/|F|$, where the infimum is taken over all subsets F of X satisfying $0 < |F| \leq |X|/2$. The well-known *Cheeger-Buser inequality* links the first non-trivial eigenvalue of the Laplacian with the Cheeger constant: $\frac{\lambda_1}{2} \leq h(X) \leq \sqrt{2k\lambda_1}$.

While any finite connected graph X has a non-zero Cheeger constant, it is rather difficult to construct a sequence of k -regular graphs of growing size such that their Cheeger constants are bounded uniformly away from zero. Given a sequence of k -regular graphs $\{X_n\}$ with $|X_n| \rightarrow \infty$, we say that $\{X_n\}$ is an *expander sequence* if there exists an $\varepsilon > 0$ such that $h(X_n) > \varepsilon$ for all n . We now give three characterizations of expanders.

Theorem 2.1. *Let $(\mathcal{G}_n)_n$ be a sequence of k -regular Cayley graphs. This sequence is an expander if one of the following equivalent statements is true:*

1. *There exists a $c > 0$ such that $h(\mathcal{G}_n) \geq c$ for every n .*
2. *There exists an $\varepsilon > 0$ such that $\lambda_1(\mathcal{G}_n) \geq \varepsilon$.*
3. *There exists a C such that for every n and every 1-Lipschitz map $\varphi: \mathcal{G}_n \rightarrow \ell^2$ we have*

$$\sum_{x, y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 \leq C|\mathcal{G}_n|^2.$$

Proof. The equivalence $1 \Leftrightarrow 2$ is due to the Cheeger-Buser inequality.

The proof of $2 \Rightarrow 3$ is based on Proposition 5.7.2 of [NY].

Set $C = \frac{k}{\varepsilon}$. Now for any n we can take $v_0 = 1_{\mathcal{G}_n}, v_1, \dots, v_{|\mathcal{G}_n|-1}$ to be the eigenvectors of the Laplacian Δ_n on \mathcal{G}_n .

Let $f: \mathcal{G}_n \rightarrow \mathbb{R}$ such that $\sum_{x \in \mathcal{G}_n} f(x) = 0$, we can write $f = a_1 v_1 + \dots + a_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1}$. Using that $\langle v_i, v_j \rangle = 0$ if $i \neq j$ we can make the following computations:

$$\begin{aligned}
\sum_{d(x,y)=1} |f(x) - f(y)|^2 &= \sum_{d(x,y)=1} f(x)(f(x) - f(y)) - f(y)(f(x) - f(y)) \\
&= \sum_{d(x,y)=1} f(x)(f(x) - f(y)) + f(x)(f(x) - f(y)) \\
&= \sum_{d(x,y)=1} 2f(x)(f(x) - f(y)) \\
&= \sum_{x \in \mathcal{G}_n} 2f(x)(\Delta_n(f)(x)) \\
&= 2\langle f, \Delta_n(f) \rangle \\
&= 2\langle a_1 v_1 + \dots + a_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1}, a_1 \lambda_1 v_1 + \dots + a_{|\mathcal{G}_n|-1} \lambda_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1} \rangle \\
&= 2\left(\lambda_1 \|a_1 v_1\|^2 + \dots + \lambda_{|\mathcal{G}_n|-1} \|a_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1}\|^2\right) \\
&\geq 2\lambda_1 \|f\|^2 \\
&= 2\lambda_1(\mathcal{G}_n) \sum_{x \in \mathcal{G}_n} |f(x)|^2.
\end{aligned}$$

Now let $\varphi: \mathcal{G}_n \rightarrow \ell^2$ be a 1-Lipschitz map. Without loss of generality we may assume that $\sum_{x \in \mathcal{G}_n} \varphi(x) = 0$. We can decompose φ according to an orthonormal basis. Using this decomposition

$$\text{we find that } 2\lambda_1(\mathcal{G}_n) \sum_{x \in \mathcal{G}_n} \|\varphi(x)\|^2 \leq \sum_{d(x,y)=1} \|\varphi(x) - \varphi(y)\|^2 \leq \sum_{d(x,y)=1} 1 \leq k|\mathcal{G}_n|.$$

Now we can bound $\sum_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2$ as follows:

$$\begin{aligned}
\sum_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 &= \sum_{x,y \in \mathcal{G}_n} \|\varphi(x)\|^2 + \|\varphi(y)\|^2 - 2\langle \varphi(x), \varphi(y) \rangle \\
&= \sum_{x \in \mathcal{G}_n} 2|\mathcal{G}_n| \|\varphi(x)\|^2 - 2\left\langle \sum_{x \in \mathcal{G}_n} \varphi(x), \sum_{y \in \mathcal{G}_n} \varphi(y) \right\rangle \\
&\leq \frac{k|\mathcal{G}_n|}{\lambda_1(\mathcal{G}_n)} |\mathcal{G}_n| \\
&\leq C|\mathcal{G}_n|^2.
\end{aligned}$$

This proves that $2 \Rightarrow 3$.

Now we only have to prove that $3 \Rightarrow 2$. Set $\varepsilon = \frac{1}{C}$ and suppose that $\lambda_1(\mathcal{G}_n) < \varepsilon$ for some n . Let f be the eigenvector v_1 for this n . Set $B = \sum_{d(x,y)=1} |f(x) - f(y)|^2$. Now we can take

$\varphi: \mathcal{G}_n \rightarrow \ell^2(\mathcal{G}_n)$ with $\varphi(x): \mathcal{G}_n \rightarrow \mathbb{R}: y \rightarrow \frac{1}{\sqrt{B}}f(y^{-1}x)$. Note that \mathcal{G}_n is a Cayley graph, therefore $y^{-1}x$ is well-defined. Now φ is 1-Lipschitz because for every $x, y \in \mathcal{G}_n$ with $d(x, y) = 1$ we have

$$\|\varphi(x) - \varphi(y)\|^2 \leq \frac{1}{B} \sum_{z \in \mathcal{G}_n} |f(z^{-1}x) - f(z^{-1}y)|^2 \leq \frac{1}{B} \sum_{d(x', y')=1} |f(x') - f(y')|^2 = 1.$$

Showing that $2\varepsilon \sum_{x \in \mathcal{G}_n} |f(x)|^2 > B$ would show that φ does not satisfy $\sum_{x, y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 \leq C|\mathcal{G}_n|^2$, because of the following argument:

$$\begin{aligned} \sum_{x, y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 &= \sum_{x, y \in \mathcal{G}_n} \|\varphi(x)\|^2 + \|\varphi(y)\|^2 - 2\langle \varphi(x), \varphi(y) \rangle \\ &= \sum_{x \in \mathcal{G}_n} 2|\mathcal{G}_n| \|\varphi(x)\|^2 - 2 \left\langle \sum_{x \in \mathcal{G}_n} \varphi(x), \sum_{y \in \mathcal{G}_n} \varphi(y) \right\rangle. \end{aligned}$$

But we have the following computation

$$\left(\sum_{x \in \mathcal{G}_n} \varphi(x) \right) (z) = \sum_{x \in \mathcal{G}_n} \frac{1}{\sqrt{B}} f(z^{-1}x) = \sum_{y \in \mathcal{G}_n} \frac{1}{\sqrt{B}} f(y) = \frac{1}{\sqrt{B}} \langle f, 1_{\mathcal{G}_n} \rangle = 0$$

so that we have

$$\left\langle \sum_{x \in \mathcal{G}_n} \varphi(x), \sum_{y \in \mathcal{G}_n} \varphi(y) \right\rangle = 0$$

and thus

$$\begin{aligned} \sum_{x, y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 &= \sum_{x \in \mathcal{G}_n} 2|\mathcal{G}_n| \|\varphi(x)\|^2 \\ &= \frac{2}{B} |\mathcal{G}_n| \sum_{x, y \in \mathcal{G}_n} |f(y^{-1}x)|^2 \\ &= \frac{2}{B} |\mathcal{G}_n|^2 \sum_{y \in \mathcal{G}_n} |f(y)|^2 \\ &\geq C|\mathcal{G}_n|^2. \end{aligned}$$

To show that $2\varepsilon \sum_{x \in \mathcal{G}_n} |f(x)|^2 > B$ we make the following computations:

$$\begin{aligned}
B &= \sum_{d(x,y)=1} |f(x) - f(y)|^2 \\
&= \sum_{d(x,y)=1} f(x)(f(x) - f(y)) - f(y)(f(x) - f(y)) \\
&= \sum_{d(x,y)=1} f(x)(f(x) - f(y)) + f(x)(f(x) - f(y)) \\
&= \sum_{d(x,y)=1} 2f(x)(f(x) - f(y)) \\
&= 2 \sum_{x \in \mathcal{G}_n} f(x)(\Delta_n(f)(x)) \\
&= 2 \sum_{x \in \mathcal{G}_n} \lambda_1(\mathcal{G}_n) f(x) f(x) \\
&= 2\lambda_1(\mathcal{G}_n) \sum_{x \in \mathcal{G}_n} |f(x)|^2 \\
&< 2\varepsilon \sum_{x \in \mathcal{G}_n} |f(x)|^2.
\end{aligned}$$

This concludes the proof. \square

The following definition of Tessera [Tes] was introduced in order to characterize the failure to embed into a Hilbert space.

Definition 2.2. Let $(\mathcal{G}_n)_n$ be a sequence of graphs. This sequence is said to be a *generalized expander* if there exists a sequence r_n with $r_n \rightarrow \infty$ as $n \rightarrow \infty$, a sequence of probability measures μ_n on $\mathcal{G}_n \times \mathcal{G}_n$ and a constant $C > 0$ such that for every 1-Lipschitz map $\varphi: (\mathcal{G}_n)_n \rightarrow \ell^2$ we have the following condition:

$$\sum_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 \mu_n(x,y) \leq C.$$

In particular, expanders in the usual sense (as above) are generalized expanders. It is proved in [Tes] that a metric space does not embed coarsely into a Hilbert space if and only if it contains a coarsely embedded sequence of expanders. In [AT], Arzhantseva and Tessera define the notions of expansion relative to subgroups, partitions, and measures, to differentiate between different cases of generalized expansion, and give examples of box spaces which do not embed into a Hilbert space and do not contain coarsely embedded expanders. We now give a natural way to find generalized expanders, which coincides with the special case of expansion relative to subgroups.

Proposition 2.3. Let r_n be a sequence such that $r_n \rightarrow \infty$ as $n \rightarrow \infty$. Let G_n be a sequence of finite k -generated groups with their corresponding Cayley graphs, and let H_n be a sequence of quotient groups of G_n with the induced metrics such that the kernel N_n of $G_n \rightarrow H_n$ is non-trivial, but $B_{G_n}(e, r_n) \cap N_n = \{e\}$.

If there exists a constant $\varepsilon > 0$ such that for every eigenvector of the Laplacian Δ_n on G_n that is not the lift of an eigenvector of the Laplacian of H_n , the corresponding eigenvalue is bigger than ε . Then the Cayley graphs of G_n form a generalized expander.

Proof. Take $D = |G_n|(|N_n| - 1)$ and take μ such that $\mu(x, y)$ is equal to $\frac{1}{D}$ if $x^{-1}y$ lies in $N_n \setminus \{e\}$ and 0 otherwise. Take $C = \frac{2k}{\varepsilon}$.

Now for any n we can take $v_0 = 1_{G_n}, v_1, \dots, v_{|H_n|-1}$ to be the lifts eigenvectors of the Laplacian on H_n ; one can easily check that these are eigenvectors of the Laplacian Δ_n on G_n . Let $w_{|H_n|}, w_{|H_n|+1}, \dots, w_{|G_n|-1}$ be the remaining eigenvectors of Δ_n on G_n . Note that since Δ_n is a symmetric operator, these eigenvectors are orthogonal. Let λ_i , with $i = 0, \dots, |G_n| - 1$, denote the corresponding eigenvalues.

Let $f: G_n \rightarrow \mathbb{R}$, we can write $f = a_0 v_0 + a_1 v_1 + \dots + a_{|H_n|-1} v_{|H_n|-1} + b_{|H_n|} w_{|H_n|} + \dots + b_{|G_n|-1} w_{|G_n|-1}$. Note that $a_0 v_0 + a_1 v_1 + \dots + a_{|H_n|-1} v_{|H_n|-1}$ is the orthogonal projection of f onto the span of $\{v_0, \dots, v_{|H_n|-1}\}$. Therefore its value at $x \in G_n$ is equal to $\frac{1}{|N_n|} \sum_{z \in N_n} f(xz)$. Using that $\langle v_i, v_j \rangle = 0$ if $i \neq j$ and $\langle v_i, w_j \rangle = 0$ for every i and j , we can make the following computations:

$$\begin{aligned}
\sum_{d(x,y)=1} |f(x) - f(y)|^2 &= \sum_{d(x,y)=1} f(x)(f(x) - f(y)) - f(y)(f(x) - f(y)) \\
&= \sum_{d(x,y)=1} 2f(x)(f(x) - f(y)) \\
&= \sum_{x \in G_n} 2f(x)(\Delta_n(f)(x)) \\
&= 2\langle f, \Delta_n(f) \rangle \\
&= 2\langle a_0 v_0 + a_1 v_1 + \dots + a_{|H_n|-1} v_{|H_n|-1} + b_{|H_n|} w_{|H_n|} + \dots \\
&\quad + b_{|G_n|-1} w_{|G_n|-1}, a_1 \lambda_1 v_1 + \dots + a_{|H_n|-1} \lambda_{|H_n|-1} v_{|H_n|-1} \\
&\quad + b_{|H_n|} \lambda_{|H_n|} w_{|H_n|} + \dots + b_{|G_n|-1} \lambda_{|G_n|-1} w_{|G_n|-1} \rangle \\
&= 2\left(0 \cdot \|a_0 v_0\|^2 + \lambda_1 \|a_1 v_1\|^2 + \dots + \lambda_{|H_n|-1} \|a_{|H_n|-1} v_{|H_n|-1}\|^2 \right. \\
&\quad \left. + \lambda_{|H_n|} \|b_{|H_n|} w_{|H_n|}\|^2 + \dots + \lambda_{|G_n|-1} \|b_{|G_n|-1} w_{|G_n|-1}\|^2\right) \\
&\geq 2\varepsilon \left(\|b_{|H_n|} w_{|H_n|}\|^2 + \dots + \|b_{|G_n|-1} w_{|G_n|-1}\|^2 \right) \\
&= 2\varepsilon \langle b_{|H_n|} w_{|H_n|} + \dots + b_{|G_n|-1} w_{|G_n|-1}, \\
&\quad b_{|H_n|} w_{|H_n|} + \dots + b_{|G_n|-1} w_{|G_n|-1} \rangle \\
&= 2\varepsilon \langle b_{|H_n|} w_{|H_n|} + \dots + b_{|G_n|-1} w_{|G_n|-1}, \\
&\quad a_0 v_0 + \dots + a_{|H_n|-1} v_{|H_n|-1} + b_{|H_n|} w_{|H_n|} + \dots + b_{|G_n|-1} w_{|G_n|-1} \rangle \\
&= 2\varepsilon \langle f - a_0 v_0 - \dots - a_{|H_n|-1} v_{|H_n|-1}, f \rangle \\
&= 2\varepsilon \sum_{x \in G_n} f(x) \left(f(x) - \frac{1}{|N_n|} \sum_{z \in N_n} f(xz) \right).
\end{aligned}$$

Now let $\varphi: \mathcal{G}_n \rightarrow \ell^2$ be a 1-Lipschitz map. We can decompose φ according to an orthonormal basis. Using this decomposition we find the following:

$$2\varepsilon \sum_{x \in G_n} \left\langle \varphi(x), \varphi(x) - \frac{1}{|N_n|} \sum_{z \in N_n} \varphi(xz) \right\rangle \leq \sum_{d(x,y)=1} \|\varphi(x) - \varphi(y)\|^2 \leq \sum_{d(x,y)=1} 1 \leq k|G_n|.$$

Now we can bound $\sum_{x,y \in G_n} \|\varphi(x) - \varphi(y)\|^2$ as follows:

$$\begin{aligned}
\sum_{x,y \in G_n} \|\varphi(x) - \varphi(y)\|^2 \mu(x,y) &= 2 \sum_{x,y \in G_n} (\|\varphi(x)\|^2 - \langle \varphi(x), \varphi(y) \rangle) \mu(x,y) \\
&= \frac{2}{D} \sum_{x \in G_n} \sum_{z \in N_n \setminus \{e\}} (\|\varphi(x)\|^2 - \langle \varphi(x), \varphi(xz) \rangle) \\
&= \frac{2}{D} \sum_{x \in G_n} \sum_{z \in N_n} (\|\varphi(x)\|^2 - \langle \varphi(x), \varphi(xz) \rangle) \\
&\leq \frac{2}{D} \sum_{x \in G_n} \left(|N_n| \|\varphi(x)\|^2 - \sum_{z \in N_n} \langle \varphi(x), \varphi(xz) \rangle \right) \\
&\leq \frac{2|N_n|}{D} \sum_{x \in G_n} \left(\|\varphi(x)\|^2 - \left\langle \varphi(x), \frac{1}{|N_n|} \sum_{z \in N_n} \varphi(xz) \right\rangle \right) \\
&\leq \frac{2|N_n|}{D} \cdot \frac{k|G_n|}{2\varepsilon} \\
&\leq \frac{2k}{\varepsilon} \\
&= C.
\end{aligned}$$

Note that the second-to-last inequality follows from the inequality $\frac{|N_n| \cdot |G_n|}{D} \leq \frac{|N_n|}{|N_n| - 1} \leq 2$. Therefore we can conclude that G_n is a generalized expander. \square

2.2 Expanders and finite covers

A result of [AT] states that given a sequence of short exact sequences of finite groups $\{N_n \rightarrow G_n \rightarrow Q_n\}_n$ such that the quotient groups $\{Q_n\}$ and the subgroups $\{N_n\}$ coarsely embed into a Hilbert space (with respect to the induced metrics from $\{G_n\}$), the sequence $\{G_n\}$ cannot contain coarsely embedded expanders. We now show that the assumption on the subgroups can be removed if the sequences $\{Q_n\}$ and $\{G_n\}$ both approximate the same group, i.e. if they are both box spaces of the same infinite group.

Proposition 2.4. *Let G be a finitely generated, residually finite group with a filtration $\{N_i\}$, and let $\{M_i\}$ be another sequence of finite index normal subgroups of G such that $N_i > M_i$ for all i . If $\square_{(N_i)} G$ coarsely embeds into a Hilbert space, then $\square_{(M_i)} G$ does not contain coarsely embedded expanders.*

Proof. Suppose that $\square_{(M_i)} G$ does contain a coarsely embedded expander sequence $\{X_i\}$. Since $\square_{(N_i)} G$ embeds coarsely into a Hilbert space and each component is equipped with a word metric, up to renumbering and rescaling the norm of the Hilbert space \mathcal{H} into which $\square_{(N_i)} G$ coarsely embeds, we can assume that we have a sequence of coarse embeddings $\varphi_i : X_i \rightarrow G/M_i$ and $\psi : \square_{(N_i)} G \rightarrow \mathcal{H}$ satisfying

$$\begin{aligned}
d_{X_i}(x, y) &\leq d_{\square_{(M_i)} G}(\varphi_i(x), \varphi_i(y)) \leq K d_{X_i}(x, y), \\
d_{\square_{(N_i)} G}(g, h) &\leq \|\psi(g) - \psi(h)\| \leq \frac{1}{K} d_{\square_{(N_i)} G}(g, h)
\end{aligned}$$

for all $x, y \in X_i$ and $g, h \in \square_{(N_i)}G$.

Let $\pi_i : G/M_i \rightarrow G/N_i$ be the quotient maps. By considering the composition of the maps $\psi \circ \pi_i \circ \varphi_i$, we obtain a 1-Lipschitz map $X_i \rightarrow \mathcal{H}$. By the first half of the argument from the proof of Proposition 2 of [AT], for each i there exists a subset $A_i \subset X_i$ satisfying $|A_i| \geq |X_i|/2$ which maps via $\pi_i \circ \varphi_i$ into some ball of radius r in G/N_i , where r depends only on the control function ρ , the Cheeger constant of the expander sequence $\{X_i\}$, and the upper bound D for the degrees of the X_i .

Now since both $\square_{(M_i)}G$ and $\square_{(N_i)}G$ are box spaces of G and G/N_i is a quotient of G/M_i , for all R there is some $m(R)$ such that for all $i \geq m(R)$, the balls of radius R in G/M_i and G/N_i are isometric to balls of radius R in G ; moreover, the quotient map $\pi_i : G/M_i \rightarrow G/N_i$ is an isometry when restricted to a ball of radius R .

Given $s \gg r$ sufficiently large (to be specified at the end of the proof), consider the quotients G/N_i for $i \geq m(r+s)$. The images of the A_i under $\pi_i \circ \varphi_i$ are contained in a ball of radius r in G/N_i , and so looking at $\varphi_i(A_i)$ in G/M_i , we see that for any element $g \in \varphi_i(A_i)$, there are at most $|B_G(r)|$ elements of $\varphi_i(A_i)$ which are at distance $\leq 2r$ from g , and all remaining elements of $\varphi_i(A_i)$ are at distance $> s$ from g so that any ball of radius s contains at most $|B_G(r)|$ elements of $\varphi_i(A_i)$. Informally, this means that elements of $\varphi_i(A_i)$ in G/M_i are grouped together in subsets of size at most $|B_G(r)|$ which are “close” together, and these small subsets are “far apart” from each other.

Due to the conditions on the map φ_i , we have the implications:

- $d_{X_i}(x, y) \leq sK^{-1} \Rightarrow d_{\square_{(M_i)}G}(\varphi_i(x), \varphi_i(y)) \leq s$,
- $d_{X_i}(x, y) > \rho^{-1}(2r) \Rightarrow d_{\square_{(M_i)}G}(\varphi_i(x), \varphi_i(y)) > 2r$.

This in turn means that in X_i , given an element $a \in A_i$, the ball of radius sK^{-1} about a contains at most $L \cdot |B_G(r)|$ elements of A_i (where L denotes the maximum cardinality of a ball of radius $\rho^{-1}(0)$ in X_i , a number that is uniformly bounded by a function depending only on D and ρ), and there are no elements of A_i in the annulus between the ball of radius sK^{-1} about a and the ball of radius $\rho^{-1}(2r)$ about a . The maximum possible number of balls of radius $\rho^{-1}(2r)$ that are at least $sK^{-1} - \rho^{-1}(2r)$ apart from each other that one can have in X_i is

$$\frac{|X_i|}{\left| B_{X_i} \left(\frac{\rho^{-1}(2r) + sK^{-1}}{2} \right) \right|},$$

where $|B_{X_i}(t)|$ denotes the smallest cardinality of a ball of radius t in X_i . Thus, we have

$$\frac{|X_i|}{2} \leq |A_i| \leq \frac{|X_i| \cdot L \cdot |B_G(r)|}{\left| B_{X_i} \left(\frac{\rho^{-1}(2r) + sK^{-1}}{2} \right) \right|} \leq \frac{2|X_i| \cdot L \cdot |B_G(r)|}{\rho^{-1}(2r) + sK^{-1}}.$$

Choosing s such that $s > K(4L \cdot |B_G(r)| - \rho^{-1}(2r))$ (a condition that depends only on ρ , K , G , and uniform properties of the sequence $\{X_i\}$) contradicts the above inequalities, completing the proof. \square

3 Subgroups of the free group

To construct box spaces of the free group which do not admit a coarse embedding into a Hilbert space without containing coarsely embedded expanders, we will use two sequences of subgroups of the free group: one which gives rise to a box space which is an expander, and one which does admit a coarse embedding into a Hilbert space. We then use information about these two sequences to prove that the box space obtained using certain intersections of these subgroups has the desired properties. In the following two subsections, we will describe the two sequences of subgroups.

3.1 Constructing subgroups of F_3

In this section we will define a sequence of nested finite index normal subgroups N_n of the free group F_3 . We will rely heavily on the machinery described in [Lub] to construct a sequence of Ramanujan graphs, and will frequently refer to relevant results and proofs in [Lub].

We fix the prime $p = 5$, noting that $p \equiv 1 \pmod{4}$, and an odd prime $q \neq 5$ such that -1 is a quadratic residue modulo q and 5 is a quadratic residue modulo $2q$. Such a prime exists, for example $q = 29$.

Consider $\mathbb{H}(\mathbb{Z})$, the integer quaternions, with the equivalence relation $a \sim b$ if there exists $m, n \in \mathbb{N}$ such that $5^n a = \pm 5^m b$. Note that the equivalence relation \sim is compatible with multiplication in $\mathbb{H}(\mathbb{Z})$. Recall that the norm N on $\mathbb{H}(\mathbb{Z})$ is defined by $N(\alpha) = \alpha \bar{\alpha}$, where $\bar{\alpha}$ is the quaternion conjugate to α . Abusing the notation, we will also write α for the equivalence class of α with respect to \sim . Note that for elements $\alpha \in \mathbb{H}(\mathbb{Z})/\sim$ with $N(\alpha) = 5^m$ for some $m \in \mathbb{Z}$, we have $\alpha^{-1} = \bar{\alpha}$.

Proposition 3.1. *The subgroup $\Lambda(2)$ of $\mathbb{H}(\mathbb{Z})/\sim$ generated multiplicatively by the set $S_5 := \{1 + 2i, 1 + 2j, 1 + 2k\}$ is the free group F_3 on the set S_5 .*

Proof. This is precisely Corollary 2.1.11 of [Lub]. □

An equivalent way to see this free group is as in Section 7.4 of [Lub]. Consider the group $\Gamma = \mathbb{H}(\mathbb{Z}[\frac{1}{p}])^\times / Z(\mathbb{H}(\mathbb{Z}[\frac{1}{p}])^\times)$, where Z denotes the center. Following the notation of [Lub], we can define a sequence of subgroups of Γ by $\Gamma(N) := \ker(\Gamma \rightarrow \mathbb{H}(\mathbb{Z}[\frac{1}{p}]/N\mathbb{Z}[\frac{1}{p}])^\times / Z(\mathbb{H}(\mathbb{Z}[\frac{1}{p}]/N\mathbb{Z}[\frac{1}{p}])^\times)$. The subgroup $\Gamma(2)$ is generated by the image of the set $S_5^\pm := \{1 \pm 2i, 1 \pm 2j, 1 \pm 2k\}$ and is exactly the free group $\Lambda(2)$ above.

The following theorem of [Lub] tells us that we can construct quotients of the free group which are expanders. Recall that a *Ramanujan graph* is a k -regular graph such that all of its eigenvalues apart from 0 and possibly $2k$ lie in the interval $[k - 2\sqrt{k-1}, k + 2\sqrt{k-1}]$ (thus a family of Ramanujan graphs achieves the best possible spectral gap).

Theorem 3.2. [Theorem 7.4.3, [Lub]] Let $p \equiv 1 \pmod{4}$ be a prime, and let $N = 2M$ be an integer such that $(M, 2p) = 1$. Assume that there is $\varepsilon \in \mathbb{Z}$ such that $\varepsilon^2 \equiv -1 \pmod{M}$. Consider the set S_p^\pm of the $p+1$ solutions $x_0 + x_1i + x_2j + x_3k$ of $x_0^2 + x_1^2 + x_2^2 + x_3^2 = p$ (where $x_0 > 0$ is odd, and x_1, x_2, x_3 are even). Associate to each element $x_0 + x_1i + x_2j + x_3k$ of S_p the matrix $\begin{bmatrix} x_0 + x_1\varepsilon & x_2 + x_3\varepsilon \\ -x_2 + x_3\varepsilon & x_0 - x_1\varepsilon \end{bmatrix} \pmod{M}$ in $\text{PGL}_2(M)$. Then the image of the group generated by S_p^\pm under this map is the quotient $\Gamma(2)/\Gamma(N)$, which is isomorphic to $\text{PSL}_2(M)$ if p is a quadratic residue modulo N , and the Cayley graph of $\Gamma(2)/\Gamma(N)$ with respect to the image of S_p^\pm is a non-bipartite Ramanujan graph.

We will apply this theorem to a particular sequence of our free group $\Gamma(2)$ (to which we will from now on refer to simply as F_3), namely the sequence of subgroups $N_n := \Gamma(2q^n)$.

We have chosen q such that -1 is a quadratic residue modulo q , and now we show that it is also a quadratic residue modulo q^n for any n .

Proposition 3.3. Let q be an odd prime. For every $u \in \mathbb{Z}$ and every $n \in \mathbb{N}$, if u is a quadratic residue modulo q and u is not zero modulo q , then u is a quadratic residue modulo q^n .

Proof. By induction we may assume that there exists a number b such that $b^2 \equiv u \pmod{q^{n-1}}$. So there exists a number c such that $b^2 = u + cq^{n-1}$. Now take $a = b - tcq^{n-1}$, where t is the inverse of $2b$ modulo q ($2b$ is invertible modulo q since u is not zero modulo q). Now $a^2 = b^2 - 2btcq^{n-1} + t^2c^2q^{2n-2} \equiv u + cq^{n-1} - cq^{n-1} = u \pmod{q^n}$. \square

We note that this implies that there exists a q -adic integer ε such that $\varepsilon^2 = -1$. We can thus use this ε to define the map in Theorem 3.2 so that the maps are compatible for M equal to different powers of q .

Similarly, since we chose q so that 5 will be a quadratic residue modulo $2q$, it is also a quadratic residue modulo $2q^n$ for all n .

Proposition 3.4. Let q be an odd prime. For every $u \in \mathbb{Z}$ and every $n \in \mathbb{N}$, if u is a quadratic residue modulo $2q$ and u is not zero modulo q , then u is a quadratic residue modulo $2q^n$.

Proof. By induction, we can assume that there is a number b such that $b^2 \equiv u \pmod{2q^{n-1}}$. So there exists a number c such that $b^2 = u + 2cq^{n-1}$. Now take $a = b - tcq^{n-1}$, where t is the inverse of b modulo q (b is invertible modulo q since u is not zero modulo q). Now $a^2 = b^2 - 2btcq^{n-1} + t^2c^2q^{2n-2} \equiv u + 2cq^{n-1} - 2cq^{n-1} = u \pmod{2q^n}$. \square

Thus, the assumptions of Theorem 3.2 are satisfied, and we obtain the following.

Corollary 3.5. For any $n \in \mathbb{N}$ we have that F_3/N_n is isomorphic to $\text{PSL}_2(q^n)$.

We will now investigate the properties of certain intermediate quotients N_k/N_n . We first need the following lemma.

Lemma 3.6. Let $k, n \in \mathbb{N}$ with $0 < k \leq n \leq 2k$. Then the kernel $\ker(\text{PSL}_2(q^n) \rightarrow \text{PSL}_2(q^k))$ of reduction modulo q^k is isomorphic to $\mathbb{Z}_{q^{n-k}}^3$.

Proof. We have that $\ker(\mathrm{PSL}_2(q^n) \rightarrow \mathrm{PSL}_2(q^k))$ is equal to $\{B \in \mathrm{PSL}_2(q^n) \mid B \equiv I_2 \pmod{q^k}\}$, where I_2 denotes the 2-by-2 identity matrix. This is precisely the set of matrices of the form $\begin{bmatrix} 1 + aq^k & bq^k \\ cq^k & 1 - aq^k \end{bmatrix}$ with a, b and c in $\mathbb{Z}_{q^{n-k}}$. For every two such matrices, we find

$$\begin{bmatrix} 1 + aq^k & bq^k \\ cq^k & 1 - aq^k \end{bmatrix} \begin{bmatrix} 1 + a'q^k & b'q^k \\ c'q^k & 1 - a'q^k \end{bmatrix} \equiv \begin{bmatrix} 1 + aq^k + a'q^k & bq^k + b'q^k \\ cq^k + c'q^k & 1 - aq^k - a'q^k \end{bmatrix} \pmod{q^n}.$$

Thus we have that $\ker(\mathrm{PSL}_2(q^n) \rightarrow \mathrm{PSL}_2(q^k))$ is isomorphic to $\mathbb{Z}_{q^{n-k}}^3$. \square

Corollary 3.7. *Let $k, n \in \mathbb{N}$ with $0 < k \leq n \leq 2k$. Then N_k/N_n is isomorphic to $\mathbb{Z}_{q^{n-k}}^3$.*

Proof. The map in Theorem 3.2 which provides the isomorphism between $\Gamma(2)/\Gamma(2q^n)$ and $\mathrm{PSL}_2(q^n)$ commutes with the map of reduction modulo q^k , and thus we have that $N_k/N_n \cong \Gamma(2q^k)/\Gamma(2q^n) \cong \ker(\mathrm{PSL}_2(q^n) \rightarrow \mathrm{PSL}_2(q^k)) \cong \mathbb{Z}_{q^{n-k}}^3$. \square

Remark 3.8. *A fact that will be of direct use to us later is that, since every element of N_{n-1}/N_n can be viewed as a matrix of the form $\begin{bmatrix} 1 + cq^{n-1} & dq^{n-1} \\ fq^{n-1} & 1 - cq^{n-1} \end{bmatrix}$ with c, d and f in \mathbb{Z}_q , a generating set of N_{n-1}/N_n can be given by the matrices*

$$\begin{bmatrix} 1 + q^{n-1} & 0 \\ 0 & 1 - q^{n-1} \end{bmatrix}, \begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix}.$$

We note that for any $k < n - 1$, the matrix $\begin{bmatrix} 1 + q^{n-1} & 0 \\ 0 & 1 - q^{n-1} \end{bmatrix}$ is equivalent to the matrix

$$\begin{bmatrix} q^{2n-2} + q^{n-1} + 1 & -q^{n+k} \\ q^{2n-k-3} & -q^{n-1} + 1 \end{bmatrix},$$

which is the commutator of the matrices $\begin{bmatrix} 1 & q^{k+1} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ q^{n-k-2} & 1 \end{bmatrix}$.

3.2 Homology covers

Given a finite graph X , one can construct a covering graph \tilde{X} of X such that \tilde{X} is the cover corresponding to the quotient $\pi(X) \rightarrow \bigoplus^r \mathbb{Z}_m$ of highest rank r possible. Indeed, since $\pi(X)$ is a free group, the rank r is simply the rank of this free group.

We recall that as a graph, the cover can be viewed in the following way. First, choose a maximal spanning tree T of the graph X . Construct the Cayley graph of $\bigoplus^r \mathbb{Z}_m$ with respect to the image of the free generating set of $\pi(X)$. Note that the free generating set of $\pi(X)$ is in bijection with its image in $\bigoplus^r \mathbb{Z}_m$, and also in bijection with the edges of X not contained in the maximal tree T . Let κ be the bijection between the edges not in T and this generating set of $\bigoplus^r \mathbb{Z}_m$.

Now, replace by a copy of T each of the vertices of the Cayley graph of $\bigoplus^r \mathbb{Z}_m$ with respect to the image of the free generating set of $\pi(X)$, where the different copies of T are connected

according to how the vertices in the Cayley graph $\bigoplus^r \mathbb{Z}_m$ are connected, via the correspondence between the edges not in T and the generating set of $\bigoplus^r \mathbb{Z}_m$ (that is, if two vertices v and w in X are connected by an edge e which is not in T , then given such a vertex \tilde{v} in one of the copies of T corresponding to a vertex a of $\bigoplus^r \mathbb{Z}_m$, we connect it via an edge to a vertex \tilde{w} in the copy of T corresponding to the element $a\kappa(e)$ of $\bigoplus^r \mathbb{Z}_m$).

The covering space \tilde{X} obtained in this way is called the *m-homology cover* of X .

The situation we are interested in is as follows: we have a sequence of quotients of the free group, $\{F_3/N_i\}$, metrized using the Cayley graph metric coming from the free generating set of F_3 , and we consider the sequence of their q -homology covers, with q as in the previous subsection. By covering space theory (for details, see for example [Kh13]), the q -homology covers of the F_3/N_i are also quotients of F_3 , by the subgroups

$$\Gamma(N_i) := N_i^q[N_i, N_i]$$

where N_i^q denotes the subgroup $\langle g^q : g \in N_i \rangle$ of F_3 generated by all the q th powers of elements of N_i . Since $\Gamma(N_i) < N_i$, we have that $\cap_i N_i = \{1\}$ implies that $\cap_i \Gamma(N_i) = \{1\}$ and so we can consider the box space $\square_{\Gamma(N_i)} F_3$.

The box space of a free group corresponding to a 2-homology cover was first considered by Arzhantseva, Guentner and Špakula in [AGS], who proved that such a box space coarsely embeds into a Hilbert space, as one can construct a wall structure on it using the covering space structure.

In [Kh13], this was generalised as follows: given any $m \geq 2$ and any sequence $\{X_i\}$ of 2-connected finite graphs where the number of maximal spanning trees in X_i not containing a given edge does not depend on the edge, the sequence of \mathbb{Z}_m -homology covers of the X_i coarsely embeds into a Hilbert space (uniformly with respect to i) if $\text{girth}(X_i) \rightarrow \infty$. Note that this holds even if the sequence $\{X_i\}$ does not embed coarsely into a Hilbert space.

In particular, we have that the box space $\square_{\Gamma(N_i)} F_3$ corresponding to the q -homology covers of any box space $\square_{N_i} F_3$ of the free group embeds coarsely into a Hilbert space, even if the box space $\square_{N_i} F_3$ is an expander sequence.

We now restrict ourselves to the following setting: the sequence $\{N_i\}$ is as defined in the previous subsection, and we consider the sequence of subgroups $\{\Gamma(N_i)\}$ corresponding to the q -homology covers. We have the following relation between the sequences, which we will need in the subsequent sections.

Proposition 3.9. *Let $k, n \in \mathbb{N}$ with $0 < k < n$. Then $N_n \Gamma(N_k) = N_{k+1}$.*

Proof. We will prove this proposition by induction on $n - k$. For $n = k + 1$ we clearly have that $N_{k+1} < N_{k+1} \Gamma(N_k)$. So it suffices to show that $\Gamma(N_k) = N_k^q[N_k, N_k] < N_{k+1}$. We will in fact show that $N_k^q < N_{k+1}$ and $[N_k, N_k] < N_{k+1}$.

To see that $N_k^q < N_{k+1}$, take an element $x \in N_k < F_3$. Up to the equivalence relation \sim , we can assume that x has the form $x = 1 + aq^k + bq^k i + cq^k j + dq^k k$. Then we can make the following

computation:

$$\begin{aligned}
x^q &= (1 + aq^k + bq^k i + cq^k j + dq^k k)^q \\
&= 1 + q^{k+1}(a + bi + cj + dk) + \frac{q(q-1)}{2}q^{2k}(a + bi + cj + dk)^2 + \dots \\
&\equiv 1 \pmod{q^{k+1}}
\end{aligned}$$

and so we have that $x^q \in N_{k+1}$ and thus $N_k^q < N_{k+1}$.

We also have that $[N_k, N_k] < N_{k+1}$, since the quotient N_k/N_{k+1} is abelian by Corollary 3.7. Therefore we have $\Gamma(N_k) \subset N_{k+1}$, and this proves the proposition for $n = k + 1$.

Now by induction we may assume that $N_{n-1}\Gamma(N_k) = N_{k+1}$. As $N_n < N_{n-1}$ we have that $N_n\Gamma(N_k) < N_{k+1}$.

We have that $N_{n-1}\Gamma(N_k) > N_n\Gamma(N_k)$. It suffices now to show that $N_{n-1}\Gamma(N_k) < N_n\Gamma(N_k)$, or equivalently that $N_{n-1}\Gamma(N_k)/N_n\Gamma(N_k)$ is trivial. Due to the second isomorphism theorem we have that

$$\frac{N_{n-1}\Gamma(N_k)}{N_n\Gamma(N_k)} = \frac{N_{n-1}N_n\Gamma(N_k)}{N_n\Gamma(N_k)} \cong \frac{N_{n-1}}{N_{n-1} \cap N_n\Gamma(N_k)}.$$

Now this is a quotient of N_{n-1}/N_n since $N_{n-1} \cap N_n\Gamma(N_k) > N_n$, and is therefore isomorphic to a quotient of \mathbb{Z}_q^3 as a consequence of Corollary 3.7. So it suffices to take a generating set of N_{n-1}/N_n and show that the elements of this generating set lie in $N_n\Gamma(N_k)$ modulo N_n . This will ensure that the quotient $N_{n-1}/N_{n-1} \cap N_n\Gamma(N_k)$ of N_{n-1}/N_n is trivial.

In fact it suffices to show that the generating elements lie in $\Gamma(N_{n-2})$ modulo the subgroup N_n , since $\Gamma(N_{n-2}) < N_n\Gamma(N_k)$.

Due to Corollary 3.5 we can view N_{n-1}/N_n as a subgroup of $\text{PSL}_2(q^n)$. As in Remark 3.8, an example of a generating set of N_{n-1}/N_n is

$$\left\{ \begin{bmatrix} q^{n-1} + 1 & 0 \\ 0 & -q^{n-1} + 1 \end{bmatrix}, \begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix} \right\}.$$

Now modulo q^n ,

$$\begin{aligned}
\begin{bmatrix} q^{n-1} + 1 & 0 \\ 0 & -q^{n-1} + 1 \end{bmatrix} &\equiv \begin{bmatrix} q^{n-2} + 1 & 0 \\ 0 & -q^{n-2} + 1 \end{bmatrix}^q \\
\begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix} &\equiv \begin{bmatrix} 1 & q^{n-2} \\ 0 & 1 \end{bmatrix}^q \\
\begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 0 \\ q^{n-2} & 1 \end{bmatrix}^q.
\end{aligned}$$

Thus all elements of N_{n-1}/N_n lie in $N_n\Gamma(N_k)/N_n$ so $N_{n-1}/(N_{n-1} \cap N_n\Gamma(N_k))$ is trivial and therefore $N_n\Gamma(N_k) = N_{n-1}\Gamma(N_k) = N_{k+1}$. \square

3.3 Representation theory

The aim of this section is to study representations of the quotients $F_3/(N_n \cap \Gamma(N_k))$ for certain values of n and k .

All representations of $F_3/(N_{n-1} \cap \Gamma(N_k))$ can be lifted to representations of $F_3/(N_n \cap \Gamma(N_k))$. In this section we want to show that the dimensions of the representations of $F_3/(N_n \cap \Gamma(N_k))$ which are not such lifts grow like q^n for k fixed.¹

For $k, n \in \mathbb{N}$ with $0 < 2k \leq n$ define $B_{k,n}$ as follows:

$$B_{k,n} := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in N_k/N_n \mid a \in \mathbb{Z}_{q^n}^\times, b \in \mathbb{Z}_{q^n} \right\}.$$

Another way of stating this condition on $a \in \mathbb{Z}_{q^n}^\times$ and $b \in \mathbb{Z}_{q^n}$ is that $a \equiv 1 \pmod{q^k}$ and $b \equiv 0 \pmod{q^k}$. Note that $B_{k,n}$ is a subgroup of N_k/N_n . In fact, for every such choice of a and b , we have that $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$ is an element of N_k/N_n and we thus see that $B_{k,n}$ has order $(q^{n-k})^2 = q^{2n-2k}$.

Lemma 3.10. *Let $k, n, l \in \mathbb{N}$ with $0 < 2k \leq 2k + l \leq n$. Then every irreducible representation π of $B_{k,n}$ for which $\pi \left(\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1 \end{bmatrix} \right) = \text{Id}$ and $\pi \left(\begin{bmatrix} 1 & q^{n-l-1} \\ 0 & 1 \end{bmatrix} \right) \neq \text{Id}$ has dimension q^{n-2k-l} .*

Proof. If $n = 2k$, then $l = 0$ and due to Corollary 3.7 we know that $B_{k,n}$ is abelian. In this case all irreducible representations of $B_{k,n}$ have dimension 1, which satisfies this proposition.

For other values of k and n , we will now consider the irreducible representations of $B_{k,n}$.

Take $\omega = e^{\frac{2\pi i}{q^{n-k}}}$. As $k \geq 1$ we have that $1 + q^k$ is of order q^{n-k} in $\mathbb{Z}_{q^n}^\times$ and therefore generates $\{\alpha \equiv 1 \pmod{q^k} \mid \alpha \in \mathbb{Z}_{q^n}^\times\}$. Now for every $j \in \{0, 1, \dots, q^k - 1\}$ define $\rho_j: B_{k,n} \rightarrow \mathbb{C}$ by

$$\begin{bmatrix} (1 + q^k)^\beta & b \\ 0 & (1 + q^k)^{-\beta} \end{bmatrix} \mapsto \omega^{\beta j}.$$

For every such j with $j \not\equiv 0 \pmod{q}$ set V_j to be the finite-dimensional Hilbert space with $\{\xi_x \mid x \equiv j \pmod{q^k}, x \in \mathbb{Z}_{q^{n-k}}\}$ as orthogonal basis, where ξ_x denotes the sequence indexed by elements of $\mathbb{Z}_{q^{n-k}}$ which takes the value 1 at $x \in \mathbb{Z}_{q^{n-k}}$ and 0 elsewhere. Let π_j be the representation of $B_{k,n}$ on V_j such that

$$\pi_j \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) \xi_x = e^{\frac{2\pi i a b x}{q^n}} \xi_{a^2 x}.$$

¹Alain Valette pointed out to us that a proof of this can also be given using the Mackey machine.

Now we can calculate the characters of these representations:

$$\begin{aligned}
\chi_{\rho_j} \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) &= \rho_j \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) \\
\chi_{\pi_j} \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) &= \sum_{x \equiv j \pmod{q^k}} \left\langle \xi_x, \pi_j \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) \xi_x \right\rangle \\
&= \sum_{x \equiv j \pmod{q^k}} \left\langle \xi_x, e^{\frac{2\pi i q b x}{q^n}} \xi_{a^2 x} \right\rangle \\
&= \sum_{x \equiv j \pmod{q^k}} e^{\frac{2\pi i q b x}{q^n}} \langle \xi_x, \xi_{a^2 x} \rangle
\end{aligned}$$

Note that if $a \equiv 1 \pmod{q^k}$ and $a^2 \equiv 1 \pmod{q^{n-k}}$, then $a \equiv 1 \pmod{q^{n-k}}$. Thus, if $a \not\equiv 1 \pmod{q^{n-k}}$, then for every $x \in \mathbb{Z}_{q^{n-k}}$ we have $\langle \xi_x, \xi_{a^2 x} \rangle = 0$, so $\chi_{\pi_j} \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) = 0$. If $b \not\equiv 0 \pmod{q^{n-k}}$, then $\sum_{x \equiv j \pmod{q^k}} e^{\frac{2\pi i q b x}{q^n}} = 0$, so $\chi_{\pi_j} \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) = 0$. If $a \equiv 1 \pmod{q^{n-k}}$ and $b \equiv 0 \pmod{q^{n-k}}$, then $a^2 x \equiv x \pmod{q^{n-k}}$ and $\sum_{x \equiv j \pmod{q^k}} e^{\frac{2\pi i q b x}{q^n}} = q^{n-2k} e^{\frac{2\pi i j b}{q^n}}$. Now for every $j, j' \in \{0, \dots, q^k - 1\}$ with $j \not\equiv 0 \pmod{q}$ we can compute $\langle \chi_{\pi_j \otimes \rho_{j'}}, \chi_{\pi_{j'} \otimes \rho_{j'}} \rangle$ using the fact that $|\chi_{\rho_j}(g)| = 1$ for every $g \in B_{k,n}$:

$$\begin{aligned}
\langle \chi_{\pi_j \otimes \rho_{j'}}, \chi_{\pi_{j'} \otimes \rho_{j'}} \rangle &= \frac{1}{|B_{k,n}|} \sum_{a \equiv 1, b \equiv 0 \pmod{q^k}} \left| \chi_{\rho_j} \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) \chi_{\pi_{j'}} \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) \right|^2 \\
&= \frac{1}{q^{2n-2k}} \sum_{a \equiv 1, b \equiv 0 \pmod{q^{n-k}}} \left| q^{n-2k} e^{\frac{2\pi i j b}{q^n}} \right|^2 \\
&= \frac{1}{q^{2n-2k}} q^{2k} q^{2n-4k} \\
&= 1.
\end{aligned}$$

Varying j and j' , we find $q^{2k} - q^{2k-1}$ irreducible representations of dimension q^{n-2k} . Note that all of these representations are different.

For every irreducible representation π of $B_{k,n-1}$, we can lift this to an irreducible representation $\tilde{\pi}$ of $B_{k,n}$. We can now consider the (also irreducible and pairwise distinct) representations $\tilde{\pi} \otimes \rho_j$, for $j \in \{0, 1, \dots, q-1\}$, π running through irreducible representations of $B_{k,n-1}$. For these representations we have that $\tilde{\pi} \otimes \rho_j \left(\begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix} \right) = \text{Id}$, since the matrix $\begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}$ lies in N_{n-1} and thus is trivial in $B_{k,n-1}$.

Now we can check if we have found all irreducible representations of $B_{k,n}$:

$$\begin{aligned}
\sum_{\pi \text{ rep. of } B_{k,n}} |\chi_\pi(I_2)|^2 &= \sum_{j=0}^{q^k-1} \sum_{j' \not\equiv 0 \pmod q} \left| \chi_{\pi_j \otimes \rho_{j'}}(I_2) \right|^2 + \sum_{j=0}^{q-1} \sum_{\pi \text{ rep. of } B_{k,n-1}} |\chi_{\tilde{\pi} \otimes \rho_j}(I_2)|^2 \\
&= \sum_{j=0}^{q^k-1} \sum_{j' \not\equiv 0 \pmod q} q^{2n-4k} + \sum_{j=0}^{q-1} \sum_{\pi \text{ rep. of } B_{k,n-1}} |\chi_{\tilde{\pi}}(I_2)|^2 \\
&= (q^{2k} - q^{2k-1})q^{2n-4k} + \sum_{j=0}^{q-1} q^{2n-2k-2} \\
&= q^{2n-2k} - q^{2n-2k-1} + q^{2n-2k-1} \\
&= |B_{k,n}|.
\end{aligned}$$

Thus we have found all the irreducible representations of $B_{k,n}$.

By induction we may assume that the proposition is true for $B_{k,n-1}$. If $l = 0$, then all the irreducible representations of $B_{k,n}$ where the image of $\begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}$ is not the identity have dimension q^{n-2k} , as they are necessarily those representations not arising as $\tilde{\pi} \otimes \rho_j$ with π an irreducible representation of $B_{k,n-1}$, i.e. they are those representations of the form $\pi_j \otimes \rho_{j'}$ constructed above.

If $l > 0$, then all irreducible representations where the image of $\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1 \end{bmatrix}$ is the identity, but the image of $\begin{bmatrix} 1 & q^{n-l-1} \\ 0 & 1 \end{bmatrix}$ is not, are of the form $\tilde{\pi} \otimes \rho_j$ where $\tilde{\pi}$ is the lift of an irreducible representation π of $B_{k,n-1}$. This is because if we consider the other representations, which are of the form $\pi_j \otimes \rho_{j'}$, considering where the vector ξ_1 is mapped by $\pi_j \left(\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1 \end{bmatrix} \right)$, we see that the image of $\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1 \end{bmatrix}$ cannot be equal to the identity.

Now due to the induction hypothesis we have that the dimension of π is $q^{(n-1)-2k-(l-1)} = q^{n-2k-l}$. Now the representation $\tilde{\pi} \otimes \rho_j$ has the same dimension, which completes the proof of the theorem. \square

Proposition 3.11. *Let $k, n \in \mathbb{N}$ be such that $3k \leq n-1$, then every representation of $F_3/(N_n \cap \Gamma(N_k))$ that is not the lift of a representation of $F_3/(N_{n-1} \cap \Gamma(N_k))$ has dimension at least q^{n-3k-3} .*

Proof. First note that $\Gamma(N_k)/(N_n \cap \Gamma(N_k))$ is isomorphic to N_{k+1}/N_n :

$$\begin{aligned}
\Gamma(N_k)/(N_n \cap \Gamma(N_k)) &\cong (N_n \Gamma(N_k))/N_n \\
&\cong N_{k+1}/N_n.
\end{aligned}$$

We have used the second isomorphism theorem and Proposition 3.9. Let us call this isomorphism Ψ ,

$$\Psi : \Gamma(N_k)/(N_n \cap \Gamma(N_k)) \rightarrow N_{k+1}/N_n.$$

We can thus view N_{k+1}/N_n as a subgroup of $F_3/(N_n \cap \Gamma(N_k))$, via Ψ .

Let π be a representation of $F_3/(N_n \cap \Gamma(N_k))$ that is not the lift of a representation of $F_3/(N_{n-1} \cap \Gamma(N_k))$. This means that π is non-trivial on the kernel of the map

$$F_3/(N_n \cap \Gamma(N_k)) \rightarrow F_3/(N_{n-1} \cap \Gamma(N_k)).$$

This kernel is equal to $(N_{n-1} \cap \Gamma(N_k))/(N_n \cap \Gamma(N_k))$. Considering this kernel, we see that it is in fact isomorphic to N_{n-1}/N_n :

$$\begin{aligned} (N_{n-1} \cap \Gamma(N_k))/(N_n \cap \Gamma(N_k)) &\cong (N_{n-1} \cap \Gamma(N_k))/(N_n \cap (N_{n-1} \cap \Gamma(N_k))) \\ &\cong ((N_{n-1} \cap \Gamma(N_k))N_n)/N_n \\ &\cong (N_{n-1}N_n \cap \Gamma(N_k)N_n)/N_n \\ &\cong (N_{n-1} \cap N_{k+1})/N_n \\ &\cong N_{n-1}/N_n. \end{aligned}$$

Here, we have used the fact that the N_i are nested, the second isomorphism theorem, Proposition 3.9, and that n is sufficiently larger than k . Let us call this isomorphism Φ ,

$$\Phi : (N_{n-1} \cap \Gamma(N_k))/(N_n \cap \Gamma(N_k)) \rightarrow N_{n-1}/N_n.$$

Now the isomorphisms Ψ and Φ are compatible, in the sense that Φ is just a restriction of Ψ . This means that when we restrict the representation π to N_{k+1}/N_n (viewed as a subgroup of $F_3/(N_n \cap \Gamma(N_k))$, via Ψ), this restriction is non-trivial on N_{n-1}/N_n as π is not a lift. This implies that at least one of the following elements of N_{n-1}/N_n has an image under π that is not the identity:

$$\begin{bmatrix} 1 + q^{n-1} & 0 \\ 0 & 1 - q^{n-1} \end{bmatrix}, \begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix}.$$

This is because, as in Remark 3.8, these matrices generate N_{n-1}/N_n . The matrix $\begin{bmatrix} 1 + q^{n-1} & 0 \\ 0 & 1 - q^{n-1} \end{bmatrix}$ is equivalent to $\begin{bmatrix} q^{2n-2} + q^{n-1} + 1 & -q^{n+k} \\ q^{2n-k-3} & -q^{n-1} + 1 \end{bmatrix}$, which is the commutator of $\begin{bmatrix} 1 & q^{k+1} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ q^{n-k-2} & 1 \end{bmatrix}$, as we have seen in Remark 3.8, and so the images of both of these must be non-trivial, if the image of their commutator is non-trivial. The transpose-inverse map is an automorphism, and thus, we may assume without loss of generality that $\pi \left(\begin{bmatrix} 1 & q^{n-k-2} \\ 0 & 1 \end{bmatrix} \right) \neq \text{Id}$ (since one of the other two generators have non-trivial images, this also implies that this matrix has a non-trivial image).

Let B be the subgroup corresponding to upper triangular matrices of N_{k+1}/N_n under the isomorphism Ψ between $\Gamma(N_k)/(N_n \cap \Gamma(N_k))$ and N_{k+1}/N_n . Due to Lemma 3.10 we know that $\pi|_B$ contains a representation of dimension at least q^{n-3k-3} (considering $B_{k+1,n}$ and $l = k+1$). Thus we can conclude that π has dimension at least q^{n-3k-3} . \square

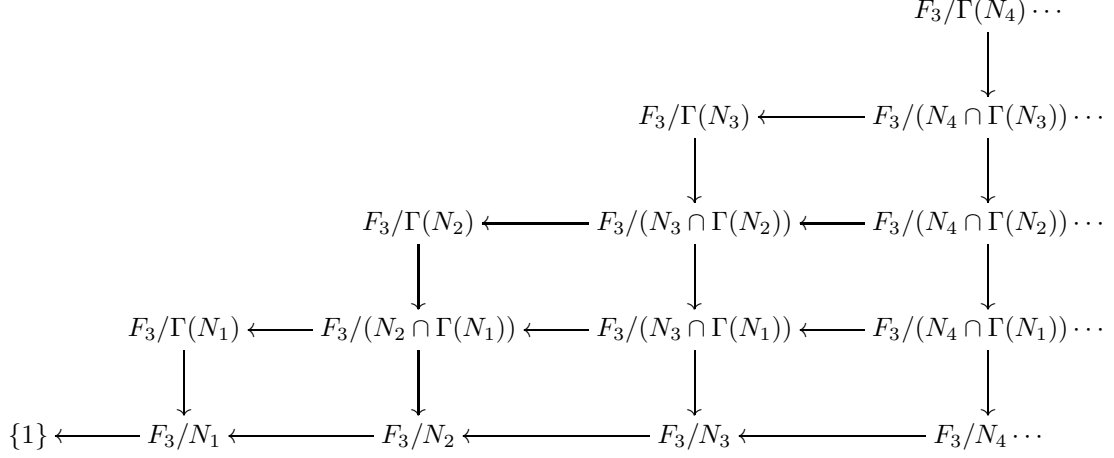


Figure 1: The magic triangle

4 Box spaces of the free group

In this section we will prove that there exist box spaces of the free group F_3 that do not embed into a Hilbert space, but do not contain coarsely-embedded expanders either. To do so we will use the following diagram, made up of quotients of the free group F_3 by intersections of the subgroups N_i with the subgroups $\Gamma(N_k)$ coming from the q -homology covers of the F_3/N_k (see Figure 1).

Note that the quotients F_3/N_i appearing along the bottom row are expanders by Corollary 3.5 and the result of Lubotzky (Theorem 3.2).

Set $f_{n,k}(m) = \#\{g \in N_n \cap \Gamma(N_k) : |g| \leq m\}$ and set $A_{n,k} = [F_3 : N_n \cap \Gamma(N_k)]$.

Lemma 4.1. *If $a^2 \equiv b^2 \pmod{q^n}$ and $q \nmid b$, then $a \equiv \pm b \pmod{q^n}$.*

Proof. We will prove this lemma by induction. For $n = 2$ the lemma follows from Exercise 1 in Section 4.3 of [DSV]. For bigger n , we have that $a^2 \equiv b^2 \pmod{q^n}$ implies $a^2 \equiv b^2 \pmod{q^{n-1}}$, so by induction we have that $a \equiv \pm b \pmod{q^{n-1}}$. Therefore there exists a $c \in \mathbb{Z}_q$ such that $a \equiv cq^{n-1} \pm b$ modulo q^n .

Now it suffices to show that $c \equiv 0 \pmod{q}$. We have that $b^2 \equiv a^2 \equiv b^2 \pm 2cbq^{n-1} \pmod{q^n}$, so $q \mid 2cb$. As q is prime, either $q \mid c$ or $q \mid 2b$. As $q \nmid b$, we have that $q \mid c$ and therefore $a \equiv \pm b \pmod{q^n}$. \square

Lemma 4.2. *For any $k, n, m \in \mathbb{N}$ with m even, we have $f_{n,k}(m) = \mathcal{O}\left(\frac{5^{\frac{13}{12}m}}{q^{3n}} + \frac{5^{\frac{7}{12}m}}{q^n}\right)$.*

Proof. Clearly it suffices to prove the theorem for $k = 0$. We have that

$$f_{n,0}(m) = \#\{\alpha \in \mathbb{H}(\mathbb{Z}) \mid [\alpha] \in N_n, N(\alpha) = 5^m\} = \#\{a + q^n(bi + cj + dk) \mid a^2 + q^{2n}(b^2 + c^2 + d^2) = 5^m\}.$$

Now $a^2 \equiv 5^m \pmod{q^{2n}}$, so due to Lemma 4.1 we have $a \equiv \pm 5^{\frac{m}{2}} \pmod{q^{2n}}$. This leaves at most $\frac{4 \cdot 5^{\frac{m}{2}}}{q^{2n}} + 2$ possibilities for a .

Now due to [DSV] we know that for any fixed $\varepsilon > 0$ we have that $\#\{(a, b, c) \mid a^2 + b^2 + c^2 = x\} = \mathcal{O}\left(x^{\frac{1}{2} + \varepsilon}\right)$. So we find a bound for $f_{n,0}(m)$:

$$\begin{aligned} f_{n,0}(m) &\leq \sum_a \mathcal{O}\left(\left(\frac{5^m - a^2}{q^{2n}}\right)^{\frac{1}{2} + \varepsilon}\right) \\ &\leq \left(\frac{4 \cdot 5^{\frac{m}{2}}}{q^{2n}} + 2\right) \mathcal{O}\left(\left(\frac{5^m}{q^{2n}}\right)^{\frac{1}{2} + \varepsilon}\right) \\ &\leq \mathcal{O}\left(\left(\frac{4 \cdot 5^{\frac{m}{2}}}{q^{2n}} + 2\right) \left(\frac{5^{m(\frac{1}{2} + \varepsilon)}}{q^n}\right)\right) \\ &= \mathcal{O}\left(\frac{5^{m(1+\varepsilon)}}{q^{3n}} + \frac{5^{m(\frac{1}{2} + \varepsilon)}}{q^n}\right). \end{aligned}$$

Now for $\varepsilon = \frac{1}{12}$ we find $f_{n,k}(m) = \mathcal{O}\left(\frac{5^{\frac{13}{12}m}}{q^{3n}} + \frac{5^{\frac{7}{12}m}}{q^n}\right)$.

□

Theorem 4.3. *There exists $N > 0$ such that for every $k, n \in \mathbb{N}$ with $n \geq N$, $18 < 18(k+1) \leq n$ and $A_{n,k} \leq q^{\frac{19}{6}n}$, we have that every eigenvalue λ of the adjacency operator A of $F_3/(N_n \cap \Gamma(N_k))$ such that some corresponding eigenvector is not the lift of an eigenvector of the adjacency operator of $F_3/(N_{n-1} \cap \Gamma(N_k))$ satisfies $\lambda \leq 5^{\frac{7}{12}} + 5^{\frac{1}{12}} < 6$.*

Proof. Without loss of generality we may assume that $\lambda \geq 2\sqrt{5}$. Take θ_j such that $\mu_j = 2\sqrt{5}\cos(\theta_j)$, where μ_j are the eigenvalues of the adjacency operator A . Due to the results of Section 4.4 of [DSV] we have

$$f_{n,k}(m) \geq \frac{1}{A_{n,k}} 5^{\frac{m}{2}} \sum_{j=0}^{A_{n,k}-1} \frac{\sin(m+1)\theta_j}{\sin \theta_j}.$$

Take $\psi_j = i\theta_j$. If $|\mu_j| \leq 2\sqrt{5}$, then θ_j is real and $\left|\frac{\sin(m+1)\theta_j}{\sin \theta_j}\right| \leq (m+1)$, and if $|\mu_j| \geq 2\sqrt{5}$, then ψ_j is real and $\frac{\sin(m+1)\theta_j}{\sin \theta_j} = \frac{\sinh(m+1)\psi_j}{\sinh \psi_j} \geq 0$. So we find the following inequality for any l , and in particular for $\mu_l = \lambda$:

$$\frac{A_{n,k}}{5^{\frac{m}{2}}} f_{n,k}(m) \geq \sum_{j=0}^{A_{n,k}-1} \frac{\sin(m+1)\theta_j}{\sin \theta_j} \geq M(\lambda) \frac{\sinh(m+1)\psi_l}{\sinh \psi_l} - (m+1)A_{n,k},$$

where $M(\lambda)$ denotes the multiplicity of the eigenvalue λ . When we take m to be the biggest even integer such that $5^{\frac{m}{2}} \leq q^{3n}$, we can use Lemma 4.2 and the fact that we chose $A_{n,k} \leq q^{\frac{19}{6}n}$ to

obtain the following:

$$\begin{aligned}
(m+1)A_{n,k} + \frac{A_{n,k}}{5^{\frac{m}{2}}} f_{n,k}(m) &\leq q^{\frac{19}{6}n} \left(m+1 + \mathcal{O} \left(\frac{5^{\frac{7}{12}m}}{q^{3n}} + \frac{5^{\frac{1}{12}m}}{q^n} \right) \right) \\
&\leq q^{\frac{19}{6}n} \left(6n \log_5(q) + 1 + \mathcal{O} \left(\frac{q^{\frac{7}{2}n}}{q^{3n}} + \frac{q^{\frac{1}{2}n}}{q^n} \right) \right) \\
&\leq q^{\frac{19}{6}n} \left(6n \log_5(q) + 1 + \mathcal{O} \left(q^{\frac{1}{2}n} + q^{\frac{-1}{2}n} \right) \right) \\
&= \mathcal{O} \left(q^{\frac{22}{6}n} \right).
\end{aligned}$$

Let V_λ be the eigenspace of A corresponding to λ on $F_3/N_n \cap \Gamma(N_k)$; since some eigenvector is not a lift from $F_3/N_{n-1} \cap \Gamma(N_k)$, the representation of $F_3/N_n \cap \Gamma(N_k)$ on V_λ is not a lift from a representation of $F_3/N_{n-1} \cap \Gamma(N_k)$. Since the eigenspace V_λ is a representation space of the group $F_3/N_n \cap \Gamma(N_k)$ (see for example [DSV]), we thus have $M(\lambda) \geq q^{n-3k-3}$ due to Proposition 3.11.

We also have

$$\frac{\sinh(m+1)\psi_l}{\sinh \psi_l} \geq \frac{e^{(m+1)|\psi_l|}}{e^{|\psi_l|}} > e^{(6n \log_5(q)-2)|\psi_l|} = \frac{q^{\frac{6n}{\log(5)}|\psi_l|}}{e^{-2|\psi_l|}}.$$

We assumed $\lambda \geq 2\sqrt{5}$, so $\psi_l \geq 0$. As $e^{2\psi_l}$ is bounded by $e^{\sqrt{5}}$ we have the following:

$$q^{n-3k-3+\frac{6n}{\log(5)}\psi_l} \leq e^{\sqrt{5}} M(\lambda) \frac{\sinh(m+1)\psi_l}{\sinh \psi_l} = \mathcal{O} \left(q^{\frac{22}{6}n} \right)$$

So for big n we find $n-3k-3+\frac{6n}{\log(5)}\psi_l \leq \frac{45}{12}n$. As $18(k+1) \leq n$ we see that $n-\frac{n}{6}+\frac{6n}{\log(5)}\psi_l \leq \frac{45}{12}n$. So $\frac{6}{\log(5)}\psi_l \leq \frac{35}{12}$ and therefore $\psi_l \leq \frac{35}{72} \log(5)$. Now we can compute λ as follows:

$$\lambda = 2\sqrt{5} \cos(\theta_l) = 2\sqrt{5} \cosh(\psi_l) \leq \sqrt{5} \left(5^{\frac{35}{72}} + 5^{\frac{-35}{72}} \right) = 5^{\frac{71}{72}} + 5^{\frac{1}{72}} < 6.$$

This proves the theorem. \square

Corollary 4.4. *Let k_i and n_i be non-decreasing sequences in \mathbb{N} with n_i increasing, $18 < 18(k_i + 1) \leq n_i$ and $A_{n_i, k_i} \leq q^{\frac{19}{6}n_i}$. Then $\square_{N_{n_i} \cap \Gamma(N_{k_i})} F_3$ does not embed into a Hilbert space.*

Proof. We want to apply Proposition 2.3, so we need to check that all the hypotheses hold.

Due to Theorem 4.3 we know there exists an $N > 0$ such that for all $n_i \geq N$, for eigenvalues λ of the adjacency operator A of $F_3/(N_{n_i} \cap \Gamma(N_{k_i}))$ such that the corresponding eigenvector is not the lift of an eigenvector of the adjacency operator of $F_3/(N_{n_i-1} \cap \Gamma(N_{k_i}))$, we have that $\lambda \leq 5^{\frac{35}{36}} + 5^{\frac{1}{36}}$.

Since the Laplacian Δ is in this case equal to $6 \text{Id} - A$ we have that every non-trivial eigenvalue of the Laplacian is greater than $6 - 5^{\frac{35}{36}} - 5^{\frac{1}{36}}$. The quotients $F_3/(N_{n_i-1} \cap \Gamma(N_{k_i}))$ and $F_3/(N_{n_i} \cap$

$\Gamma(N_{k_i})$ look like F_3 (and thus like each other) on bigger and bigger balls, so there exists a sequence r_i such that $r_i \rightarrow \infty$ as $i \rightarrow \infty$ with

$$B(e, r_i) \cap \left(N_{n_i-1} \cap \Gamma(N_{k_i}) \right) / \left(N_{n_i} \cap \Gamma(N_{k_i}) \right) = \{e\},$$

where $B(e, r_i)$ denotes the ball of radius r_i about the identity in $F_3 / (N_{n_i} \cap \Gamma(N_{k_i}))$. But on the other hand, due to the isomorphism Φ given as part of the proof of Proposition 3.11, and Corollary 3.7, we have $N_{n_i-1} \cap \Gamma(N_{k_i}) \neq N_{n_i} \cap \Gamma(N_{k_i})$, since $(N_{n_i-1} \cap \Gamma(N_{k_i})) / (N_{n_i} \cap \Gamma(N_{k_i})) \cong N_{n_i-1} / N_{n_i} \cong \mathbb{Z}_q^3$.

Now Proposition 2.3 can be applied to the subsequence of $F_3 / (N_{n_i} \cap \Gamma(N_{k_i}))$ with $n_i \geq N$. So $\square_{N_{n_i} \cap \Gamma(N_{k_i})} F_3$ contains a generalized expander and therefore does not embed into a Hilbert space, by the characterization of Tessera [Tes]. \square

The Main Theorem now follows from the following result.

Theorem 4.5. *There exist increasing sequences k_i and n_i in \mathbb{N} such that $18 < 18(k_i + 1) \leq n_i$ and $A_{n_i, k_i} \leq q^{\frac{19}{6}n_i}$, and for such n_i, k_i , the box space $\square_{N_{n_i} \cap \Gamma(N_{k_i})} F_3$ does not embed into a Hilbert space, but does not contain coarsely embedded expanders.*

Proof. Let us first check that such a sequence (n_i, k_i) exists. We have, using the information we have obtained in Sections 3.1 and 3.2 about the sizes of quotients in Figure 1,

$$\begin{aligned} A_{n_i, k_i} &= |F_3 / (N_{n_i} \cap \Gamma(N_{k_i}))| \\ &\leq |F_3 / N_{n_i}| \cdot |F_3 / \Gamma(N_{k_i})| \\ &\leq |F_3 / N_{n_i}| \cdot |F_3 / N_{k_i}| \cdot |N_{k_i} / \Gamma(N_{k_i})| \\ &\leq |\mathrm{PSL}_2(q^{n_i})| \cdot |\mathrm{PSL}_2(q^{k_i})| \cdot |\mathbb{Z}_q^{2|\mathrm{PSL}_2(q^{k_i})|+1}| \\ &\leq q^4 \cdot q^{3(n_i-1)} \cdot q^4 \cdot q^{3(k_i-1)} \cdot q^{2(q^4 q^{3(k_i-1)})+1} \\ &= q^{3n_i+3k_i+3+2q^{3k_i+1}}. \end{aligned}$$

This means that we need $3k_i + 3 + 2q^{3k_i+1}$ to be less than or equal to $\frac{1}{6}n_i$ in order to satisfy $A_{n_i, k_i} \leq q^{\frac{19}{6}n_i}$. Now it is clear that for a sequence of large enough n_i we can take a sequence of k_i which will simultaneously satisfy this and the condition $18 < 18(k_i + 1) \leq n_i$. By taking subsequences if necessary, we can ensure that the sequences n_i and k_i are increasing.

Corollary 4.4 gives us the first part of the statement. For the second part of the statement, we can now apply Proposition 2.4 to the box space $\square_{N_{n_i} \cap \Gamma(N_{k_i})} F_3$ and the box space $\square_{\Gamma(N_{k_i})} F_3$, which is embeddable into Hilbert space by the main result in [Kh13] (described in Section 3.2) as it is a sequence of q -homology covers of the graphs F_3 / N_{k_i} which satisfy the necessary conditions. \square

5 Questions and remarks

- A consequence of Theorem 4.5 is that it is possible to have two box spaces of the same group with respect to *meshed* sequences of subgroups (i.e. sequences of subgroups $\{H_i\}$)

and $\{K_i\}$ with $H_1 > K_1 > H_2 > K_2 > H_3 \dots$) such that one of the box spaces coarsely embeds into a Hilbert space, and the other does not.

Indeed, after passing to a subsequence, we can find a box space $\square_{N_{n_j} \cap \Gamma(N_{k_j})} F_3$, that can be nested with the box space $\square_{\Gamma(N_{k_j})} F_3$ (this corresponds to taking subgroups which form a sequence of “steps” in our diagram of subgroup intersections). We know from [Kh13] that $\square_{\Gamma(N_{k_j})} F_3$ embeds coarsely into ℓ^2 , while $\square_{N_{n_j} \cap \Gamma(N_{k_j})} F_3$ does not by the above.

- In the constructions of groups containing expanders (see [AD, Gro, Osa]), it is necessary that the girth of the graphs that one wishes to embed via small cancellation labellings grows linearly with the diameter. It can be shown that the chosen sequence of quotients satisfies this condition.
- We have shown that one can choose a sequence in the triangle of intersections (Figure 1) which does not coarsely embed into a Hilbert space, but does not contain coarsely embedded expanders. This sequence lies on a path that lies “close enough” to the horizontal expander sequence. In an upcoming paper, the first author proves that the horizontal sequences in such a triangle (or, more generally, covers of expanders of uniformly bounded degree) form an expander sequence. What can be said of other sequences in the triangle? Is there a relationship between k_i and n_i for the quotients $F_3/(N_{n_i} \cap \Gamma(N_{k_i}))$ which guarantees embeddability into a Hilbert space? Note that in [DK], it is shown that two box spaces $\square_{N_i} G$ and $\square_{M_i} G$ with $M_i > N_i$ and $[M_i : N_i]$ uniformly bounded independently of i need not be coarsely equivalent.
- Let $\{N_i\}$ be a different sequence of subgroups of the free group which gives rise to an expander. Can one prove similar results?
- Do homology covers of quotients of non-free groups coarsely embed into a Hilbert space? If G is a finitely generated group, not necessarily free, what can one say about the box space corresponding to the inductively defined sequence of subgroups $N_1 := G$, $N_{i+1} := N_i^q[N_i, N_i]$? If this box space embeds, can one recreate the triangle argument in such a case?
- It is unknown whether there exists a bounded geometry metric space which does not coarsely embed into Hilbert space, but does coarsely embed into ℓ^p for some $p > 2$. Note that such a space cannot contain coarsely embedded expanders. Does the box space constructed in this paper embed coarsely into ℓ^p ?

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